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ON THE VOLUME OF A POLYHEDRON.

BY H. B. NEWSON.

MANY elementary text books on analytical geometry give a formula for finding the area of a plane polygon of n sides in terms of the coordinates of the n vertices. A similar formula exists for finding the volume of a polyhedron of n faces. In the present paper I present a method for deducing these formulas which I have frequently used in my classes, and find well suited for the purposes of instruction. It may be noted that this method applies not merely to ordinary convex polyhedrons, but to any solid bounded by a finite number of plane faces.

1. Area of a polygon of n sides. Let the vertices of a triangle be numbered 1, 2, 3; and let the rectangular cartesian coordinates of these vertices be denoted respectively by (x_1, y_1) , (x_2, y_2) , (x_3, y_3) . Let A denote the area of the triangle. Then by a well-known formula we have:

$$2A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \equiv \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}. \quad (1)$$

The area of the triangle is here expressed as the sum of three determinants which, so to speak, correspond to the three sides of the triangle. The determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ corresponds to the side (1, 2) taken in the sense from 1 to 2, and similarly for the other sides. Thus twice the area of the triangle is found by writing down the sum of the determinants corresponding to the sides of the triangle, taken all in the same sense around the triangle. The law of the formation of the determinants may also be expressed by the convention of a moving point describing the perimeter of the triangle.

We can now apply this result to the problem of finding the area of a polygon of n sides. Let the vertices of the polygon be numbered in order 1, 2, 3, . . . , n , and let the coordinates of the vertex k be denoted by (x_k, y_k) , ($k = 1, 2, 3, \dots, n$). Let us divide the polygon into a number of triangles, and let a moving point describe the perimeter of each triangle always in the same sense. It is evident that by this process each of the interior

lines will be described twice, but the two descriptions are always in opposite senses; while each of the bounding lines is described once, and always in the same sense. In taking the sum of all these triangles, the two determinants corresponding to the two descriptions of an interior line in opposite senses differ only in sign, and hence cancel each other. Thus we have for twice the area of the polygon, the sum of the determinants corresponding to the sides of the polygon. The formula is therefore :

$$2A = \left| \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix} \right| + \left| \begin{matrix} x_2 & y_2 \\ x_3 & y_3 \end{matrix} \right| + \left| \begin{matrix} x_3 & y_3 \\ x_4 & y_4 \end{matrix} \right| + \dots + \left| \begin{matrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{matrix} \right| + \left| \begin{matrix} x_n & y_n \\ x_1 & y_1 \end{matrix} \right|. \quad (2)$$

It may be remarked that each of the determinants in this formula represents twice the area of the triangle formed by joining the origin to two adjacent vertices of the polygon.

2. Volume of a polyhedron of n faces. In like manner if the vertices of a tetrahedron be numbered 1, 2, 3, 4; and if the corresponding coordinates be written (x_1, y_1, z_1) , etc., we know, V being the volume of the tetrahedron, that :

$$6V = \left| \begin{matrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{matrix} \right| \equiv \left| \begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix} \right| + \left| \begin{matrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_2 & y_2 & z_2 \end{matrix} \right| + \left| \begin{matrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{matrix} \right| + \left| \begin{matrix} x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \\ x_3 & y_3 & z_3 \end{matrix} \right|. \quad (3)$$

Thus we have six times the volume of the tetrahedron expressed by the sum of four determinants which may be said to correspond to the four faces of the tetrahedron. The law of the formation of these determinants may be expressed as follows: let us suppose the perimeter of each face of the tetrahedron to be described by a moving point, and all in the same direction. In determining directions in this case it is supposed that the moving point is always on the external side of the faces of the tetrahedron. The best way for the reader to realize this law is to take in his hand a wooden model of a tetrahedron whose vertices have been numbered 1, 2, 3, 4; and then with each face in turn before him write down the corresponding determinant going around the faces always in the same direction.

A polyhedron of any number of faces can always be decomposed by internal planes into a number of tetrahedrons. Let us find by the above formula the volume of each of these tetrahedrons and take their sum. We readily see that any face of one of these tetrahedrons which is not a part of the bounding

surface of the polyhedron is described twice, but each time in opposite directions, while each face which is a part of the bounding surface is described only once, and all of them in the same direction. The two determinants corresponding to the description of an internal face in opposite directions differ only in sign and hence cancel each other. We have, therefore, the following theorem :

Six times the volume of a polyhedron is equal to the sum of all the determinants corresponding to the triangles which form its bounding surface, each triangle being described in the same direction.

If one or more of the faces of the polyhedron be polygons, each such polygon must be divided into triangles.

Each of these determinants of the third order represents six times the volume of the tetrahedron formed by joining the origin to the vertices of a face triangle.

3. Area of a closed curve. Formula (2) for the area of a polygon may be expressed more concisely as follows :

$$A = \frac{1}{2} \sum \begin{vmatrix} x_k & y_k \\ x_{k+1} & y_{k+1} \end{vmatrix} \quad (k = 1, 2, 3, \dots, n).$$

We may consider a closed curve as a polygon of an infinite number of sides. The coordinates of two adjacent vertices may be expressed by (x, y) and $(x + dx, y + dy)$ and the determinant $\begin{vmatrix} x_k & y_k \\ x_{k+1} & y_{k+1} \end{vmatrix}$ reduces to $\begin{vmatrix} x & y \\ dx & dy \end{vmatrix}$. Replacing the summation sign by the sign of integration we have,

$$A = \frac{1}{2} \int_c (xdy - ydx),$$

a well-known formula for finding the area of a closed curve.

In order to complete this investigation in a symmetrical manner we should deduce from the above theorem for the volume of a polyhedron an integration formula for finding the volume of a closed surface. This I have not been able to do to my satisfaction, and I leave the problem with the readers of the ANNALS.